Complex analysis for EE, 2012-13, problem set 4

Note to students: Due to the current situation, some students have yet to cover the Cauchy-Goursat theorem in class. Although some different variants of this theorem exist, the main result is this: if Γ is a closed simple curve in $\mathbb{C}$, which is contained in a region Ω along with its interior (the section of $\mathbb{C}$ enclosed by it), and if $f : \Omega \to \mathbb{C}$ is holomorphic, then $\int_\Gamma f(z)dz = 0$.

The use of this theorem is required in some parts of this sheet. Problem 10 deals with this theorem directly, extending the results shown in class to an a priori different setting.

1. Evaluate the following line integrals:
   (a) $\int_\Gamma \bar{z}dz$ where Γ is the triangular curve connecting $0, 1+i, i$.
   (b) $\int_{|z|=2} \frac{dz}{z^2-1}$.
   (c) $\int_\Gamma \mathrm{Re}\,(z)\,dz$ where Γ is the line segment connecting $5i-7$ to $5i+9$.
   Do so both directly and by representing $\mathrm{Re}\,(z) = p(z)$ for some polynomial p, along Γ.
   (d) $\int_\Gamma \mathrm{Re}\,(z)\,dz$ where Γ is the circle of radius $r > 0$ about 0.
   Do so both directly and by representing $\mathrm{Re}\,(z) = \frac{p(z)}{q(z)}$ for some two polynomials p, q, along Γ.

2. Evaluate the line integral $\int_\gamma \frac{dz}{z}$ where
$$\gamma(t) = \begin{cases} -1+i+e^{-it} & 0 \le t < \frac{\pi}{2} \\ -1-i+e^{i(\pi-t)} & \frac{\pi}{2} \le t \le \pi \end{cases}$$

3. Evaluate the line integral $\int_\gamma \left(e^{1/\bar{z}} + \frac{1}{e^{1/\bar{z}}}\right) dz$ where $\gamma(t) = re^{it}$ for some $r > 0$, and $t \in [0, \pi]$.

4. (a) Suppose $\Omega \subseteq \mathbb{C}$ has a boundary consisting of a finite number of simple, closed, piecewise C^1 curves. Assume that $f(x+iy) = u(x,y) + iv(x,y)$ is defined in a neighborhood of $\overline{\Omega}$, and that u, v have continuous partial derivatives in that neighborhood. Using Green's theorem, express $\int_{\partial\Omega} f(z)dz$ in terms of the partial derivatives of u, v.
   Caution: f needn't be holomorphic.
   (b) Show that the area of every triangle $\Delta \subseteq \mathbb{C}$ is given by $S(\Delta) = \frac{1}{2i}\int_{\partial\Delta} \bar{z}dz$.

5. By integrating $\frac{R+z}{z(R-z)}$ along a suitable circle, show that
$$\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2} d\theta = 2\pi,$$
   for some $0 < r < R$. This will fact will come in handy in various situations.

6. As you'll recall, Weierstrass's theorem states that every continuous function $f : [a,b] \to \mathbb{R}$ can be uniformly approximated by polynomials. That is, there exist polynomials $p_n(x)$ such that $\sup_{x\in[a,b]} |f(x) - p_n(x)| \underset{n\to\infty}{\longrightarrow} 0$.
   Use integration along curves to prove that the same does not hold for $\frac{1}{z}$ when viewed as a function on $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ (which is, just as $[a,b]$ is for $\mathbb{R}$, a compact subset of the complex plain). More explicitly, find $\varepsilon > 0$ such that for every polynomial $p(z)$ it holds that $\sup_{z\in S^1} \left|\frac{1}{z} - p(z)\right| \ge \varepsilon$.
   Hint: consider closed contours about the origin.

7. In each of the following cases, explain why the given line integral vanishes:

   (a) $\int_\gamma \frac{dz}{(z-3)^3}$ where $\gamma(t) = i + 4e^{it}$ for $t \in [0, 2\pi]$.

   (b) $\int_\gamma z\,|z|\,dz$ where $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$.

   (c) $\int_\gamma \frac{e^{z^2}}{z^2+4} dz$ where $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$.

   (d) $\int_\gamma \frac{1}{\sin^2 z} dz$ where $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$.

8. Let $\Omega \subset \mathbb{C}$ be a region, and $f : \Omega \to \mathbb{C}$ a holomorphic function. Suppose $\gamma : [a, b] \to \Omega$ is a closed simple contour. We'll show that $\int_\gamma \overline{f(z)} f'(z) dz$ is purely imaginary.

   (a) Use the Cauchy-Riemann equations to express the real and imaginary parts of the integrand $\overline{f(z)} f'(z)$.

   (b) Utilize your expression to show that

   $$\mathrm{Re}\left(\int_\gamma \overline{f(z)} f'(z) dz\right) = \int_a^b \left[(u(\gamma(t))u'_x(\gamma(t)) + v(\gamma(t))v'_x(\gamma(t)))\,\mathrm{Re}\,(\gamma'(t)) + \right.$$
   $$\left. + \left(u(\gamma(t))u'_y(\gamma(t)) + v(\gamma(t))v'_y(\gamma(t))\right)\mathrm{Im}\,(\gamma'(t))\right] dt$$

   (c) Group the elements of the integrand and utilize C-R again to express it as

   $$(u \circ \gamma)(t)(u \circ \gamma)'(t) + (v \circ \gamma)(t)(v \circ \gamma)'(t).$$

   (d) Deduce the claim.

9. In this nonobligatory exercise we prove some of the properties of the *winding number*. Let $\gamma : [a, b] \to \mathbb{C}$ is a closed contour, and suppose $w \in \mathbb{C}$ isn't in the image of γ. We define the *winding number* of γ with respect to w, or the *index* of w with respect to γ, as $n(w, \gamma) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-w}$.

   (a) Show that $n(\cdot, \gamma) : (\mathbb{C} \setminus \gamma[a, b]) \to C$ is a continuous function.

   (b) Show that $n(w, \gamma)$ is always an integer. That is, that $n(\cdot, \gamma) : (\mathbb{C} \setminus \gamma[a, b]) \to \mathbb{Z}$. Hint: one way of achieving this feat is by considering the function $h(t) = \int_a^t \frac{\gamma'(\tau)}{\gamma(\tau)-w} d\tau$. Verify that it is a piecewise-differentiable continuous function, and then examine the derivative of $e^{-h(t)}(\gamma(t) - w)$. Deduce the claim from from what you discover.

   (c) Show that if w_1, w_2 are connected by a path α whose image is disjoint from γ's, then
   $$n(w_1, \gamma) = n(w_2, \gamma).$$

10. In class, we've proved a variant of the Cauchy-Goursat theorem which assumes the continuity of $f'(z)$ for some holomorphic function f. In fact, the assumption that f' is continuous is redundant (as we'll soon see, it's actually a consequence of f being holomorphic). In this exercise, we shall prove that for integration over rectangles. As in problem 9, this exercise is non-mandatory.

    (a) Let $R \subseteq \mathbb{C}$ be a rectangle. Explain why $\int_{\partial R} dz = \int_{\partial R} z dz = 0$.

    (b) Let $R \subseteq \mathbb{C}$ be a rectangle, and f a holomorphic function on a neighborhood of R. Divide R into 4 identical rectangles $R^{(j)}$ for $j = 1, \ldots, 4$. Show that there exists j such that $\left|\int_{\partial R^{(j)}} f(z)dz\right| \geq \frac{1}{4} \left|\int_{\partial R} f(z)dz\right|$.

(c) Use that fact to arrive by a sequence of rectangles $R = R_0, R_1, \dots, R_n, \dots$, each taking a quarter of the area of the preceding one, such that

$$\left| \int_{\partial R_n} f(z)dz \right| \geq \frac{1}{4^n} \left| \int_{\partial R} f(z)dz \right|.$$

Explain the existence of (a single) $z_0 \in \mathbb{C}$ such that $z_0 \in R_n$ for all n.

(d) Find some $M > 0$ such that for every $\varepsilon > 0$ you can use the existence of $f'(z_0)$, as well as the first part of this question, to come by the bound $\left| \int_{\partial R_n} f(z)dz \right| \leq \varepsilon \frac{M}{4^n}$ (for a large enough n).
Note: M must not depend on n.

(e) Deduce the claim.